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Thermodynamics of the massive Thirring–sine-Gordon model: the Bethe ansatz variational method

S G Chung† and Yia-Chung Chang

Department of Physics and Materials Research Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

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Abstract. The Bethe ansatz variational approach to the thermodynamics of the massive Thirring–sine-Gordon model in the attractive coupling regime and in the zero-change sector is discussed in detail.

1. Introduction

The massive Thirring model (MTM) is a theory of self-interacting fermions with mass m_0 in one dimension described by the Lagrangian

$$\mathscr{L} = \mathrm{i}\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi - m_0\bar{\Psi}\Psi - \tfrac{1}{2}g_0 j^{\mu}j_{\mu} \tag{1.1}$$

where $j^{\mu} = \frac{1}{2}[\bar{\Psi}, \gamma^{\mu}\Psi]$ is the fermion current and g_0 is the coupling constant. In a basis in which γ^5 is diagonal, the Hamiltonian is written as (letting $\Psi = \binom{\Psi_1}{\Psi_2}$)

$$H = \int \mathrm{d}x\, [-\mathrm{i}(\Psi_1^{\dagger}\partial_x\Psi_1 - \Psi_2^{\dagger}\partial_x\Psi_2) + m_0(\Psi_1^{\dagger}\Psi_2 + \Psi_2^{\dagger}\Psi_1) + 2g_0\Psi_1^{\dagger}\Psi_2^{\dagger}\Psi_2\Psi_1]. \tag{1.2}$$

This model attracted much interest originally as a model for describing the electron gas with local interactions. Recently, two new aspects of this model have been revealed. First, it was demonstrated that the MTM is equivalent to the spin-$\frac{1}{2}$ Heisenberg model in a continuum limit [1]. This equivalence has led to some understanding of the MTM thermodynamics [2], because the thermodynamics of the spin-$\frac{1}{2}$ Heisenberg model had been thoroughly studied by the Bethe ansatz (BA) method [3]. Second, it was also shown that the MTM is equivalent to the sine-Gordon (SG) model [4, 5]. This is an important aspect, since the SG model has such generality and simplicity that one frequently encounters the model in many subfields of physics. One of the most challenging problems in recent soliton physics has been to understand the quantum statistical mechanics of the SG model [6]. Several years ago, when the quantum inverse scattering technique was first developed for the SG system [7], people expected it to shed new light on this problem. However, the resulting BA state turns out to be so complicated that the construction of the statistical mechanics based on it seems almost impossible.

In this paper, we shall give a full description of the MTM–SG thermodynamics in the attractive regime and in the zero-charge sector. We start with the BA theory of MTM recently developed by Bergknoff and Thacker (BT) [8] and Korepin [9]. We then

† Present address: Department of Physics, Western Michigan University, Kalamazoo, MI 49008, USA.

extend their theory to study the quantisation of multi-elementary excitations in MTM. The elementary excitations in MTM include breathers, Korepin's excitations and free holes. We show that the quantisation of the physical momentum of each elementary excitation can be written in a BA form in terms of renormalised phase shifts due to scatterings with all other elementary excitations. Finally we formulate the MTM thermodynamics using the BA variational method initiated by Yang and Yang [10] and further developed by Gaudin [11] and Takahashi and Suzuki [3]. One can refer to our method as the direct BA method in order to distinguish it from the other type of BA formalism which was developed mainly by Fowler and Zotos (FZ) [2]. The latter was based on the Takahashi–Suzuki formalism originally developed for the spin-$\frac{1}{2}$ Heisenberg model thermodynamics.

One major finding of our study is that, at finite temperatures, the excitation energy of the soliton (antisoliton) is discontinuous as a function of the coupling constant, whereas the breather excitation energy and the free energy are continuous. A preliminary report of this finding was presented in [12].

The present paper is organised as follows. In the next section, we give a compact and unified description of the Hamiltonian eigenstates, physical vacuum and elementary excitations based on the studies of BT [8], Korepin [9] and the present authors [13]. In § 3, we quantise generic excited states on the physical vacuum by imposing periodic boundary conditions (PBC). We show that the interaction between elementary excitations can be described in terms of the renormalised two-body S matrices. In § 4, the MTM thermodynamics is formulated by the BA variational method and the finite-temperature excitation spectrum is discussed. The latter half of § 4 is devoted to some analysis of the basic equations obtained regarding the discontinuity of soliton mass at finite temperatures and the unphysical nature of Korepin's excitations. A summary and some concluding remarks are given in the final section.

After the completion of the present paper, we have received strong criticism that the soliton mass cannot be discontinuous. Several authors [20] proposed a definition of soliton mass which leads to a continuous soliton mass. They disputed that our equation (4.15) is a definition of soliton mass, saying that it is a wrong definition *because it leads to an unphysical discontinuity in the soliton mass.* We do not agree with these authors, but the reader should be referred to [20] and critically examine our arguments in the paper.

2. Hamiltonian eigenstates, physical vacuum and elementary excitations

We start with the Bethe ansatz wavefunction of BT:

$$|\beta_1 \dots \beta_N\rangle = \int \mathrm{d}x_1 \dots \int \mathrm{d}x_N \, \chi(\beta_1 \dots \beta_N, x) \prod_{i=1}^{N} A^{\dagger}(\beta_i, x_i)|0\rangle \tag{2.1}$$

with

$$\chi(\beta_1 \dots \beta_N, x) = \exp\left(\mathrm{i} \sum_{j=1}^{N} m_0 \sinh \beta_j x_j\right) \prod_{i<j} [1 - \tfrac{1}{2}\mathrm{i}g \tanh \tfrac{1}{2}(\beta_i - \beta_j)\varepsilon(x_i - x_j)] \tag{2.2}$$

where the β are rapidity variables,

$$\varepsilon(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \end{cases}$$

and

$$A^{\dagger}(\beta, x) = \cos\theta(\beta)\Psi_1^{\dagger}(x) + \sin\theta(\beta)\Psi_2^{\dagger}(x) \tag{2.3}$$

with $\cot 2\theta(\beta) = \sinh\beta$. One can easily see that (2.1) is an eigenstate of the Hamiltonian (1.2) with energy

$$E = \sum_{i=1}^{N} m_0 \cosh\beta_i. \tag{2.4}$$

Equation (2.1) is also an eigenstate of the momentum operator

$$G = -\mathrm{i}\int \mathrm{d}x(\Psi_1^{\dagger}\partial_x\Psi_1 + \Psi_2^{\dagger}\partial_x\Psi_2) \tag{2.5}$$

with the momentum

$$P = \sum_{i=1}^{N} m_0 \sinh\beta_i. \tag{2.6}$$

It is seen from (2.1) that the two-body S matrix for the scattering of two *bare* excitations with rapidities β_i and β_j is

$$S_{ij} = \frac{1 - \frac{1}{2}\mathrm{i}g_0 \tanh[(\beta_i - \beta_j)/2]}{1 + \frac{1}{2}\mathrm{i}g_0 \tanh[(\beta_i - \beta_j)/2]} \equiv \exp[\mathrm{i}\phi(\beta_i - \beta_j)] \tag{2.7}$$

where $\phi(\beta)$ is the two-body phase shift:

$$\phi(\beta) = -\mathrm{i}\ln\left(-\frac{\sinh(\beta/2 - \mu\mathrm{i})}{\sinh(\beta/2 + \mu\mathrm{i})}\right) = 2\tan^{-1}[\cot\mu \tanh(\beta/2)] \tag{2.8}$$

and where $\mu = -\cot^{-1}(g_0/2)$. Note that $g_0 > 0$ (repulsive interaction), $g_0 = 0$ (free fermion theory) and $g_0 < 0$ (attractive interaction) correspond, respectively, to $\mu < \pi/2$, $\mu = \pi/2$ and $\mu > \pi/2$. In this paper we restrict ourselves to the attractive case $\mu > \pi/2$.

The bare elementary excitations described by (2.1) were studied by BT and Korepin. The simplest of these is the Dirac sea mode $\beta = \alpha + \mathrm{i}\pi$, where α is real. This mode has a negative energy. Filling up the Dirac sea modes gives a physical vacuum (see below). The elementary excitations which have normalisable wavefunctions are generally called 'n-strings' because they are represented by a vertical array of n points in the complex rapidity plane [8, 9]:

$$\beta_{l_n} = \alpha_n + \mathrm{i}\pi B + \mathrm{i}\omega l \qquad l_n = -(n-1), -(n-3), \ldots, (n-1) \tag{2.9}$$

where α_n is real, $B = 0$ or 1 and $\omega = \pi - \mu$. The integer n in (2.9) is not arbitrary. The conditions for allowed n are

$$\sin p\omega \sin(n-p)\omega \begin{cases} > 0 & \text{if } B = 0 \\ < 0 & \text{if } B = 1 \end{cases} \qquad \text{for } p = 1, 2, \ldots, n-1. \tag{2.10}$$

Since $0 < \omega < \pi$, one can show that (2.10) is equivalent to

$$\left[\frac{p\omega}{\pi}\right] + \left[(n-p)\frac{\omega}{\pi}\right] = \left[(n-1)\frac{\omega}{\pi}\right] \qquad \text{for } p = 1, 2, \ldots, n-1 \tag{2.11}$$

where [ ] is the Gauss symbol. The solution of (2.11) can be found in [3]. The method is to express the number $P_0 = \pi/\omega$ as a continued fraction:

$$P_0 = \nu_1 + \frac{1}{\nu_2 + \ddots}. \tag{2.12}$$

where $\nu_1, \nu_2, \ldots$, are integers $\geqslant 2$. Note that $P_0 = 2$ corresponds to $\mu = \pi/2$, the free fermion theory. For each step $\nu_1, \nu_2, \ldots$, of the continued fraction, there exists a corresponding set of integers n [3].

In this paper, we examine the case $P_0 = \nu_1 + 1/\nu_2$. As will become clear below, by studying this case we can understand essential aspects of the MTM thermodynamics in the attractive coupling regime. In this case, the allowed strings are

$$n = \begin{cases} 1, 2, \ldots, \nu_1 \text{ and } 1 + j\nu_1 & (j = \text{odd} \leqslant \nu_2) \quad \text{for } B = 0 \\ 1 + j\nu_1 & (j = \text{even} \leqslant \nu_2) \quad \text{for } B = 1. \end{cases} \tag{2.13}$$

The allowed strings for the simple case $\nu_1 = 2$ and $\nu_2 = 3$ are illustrated in figure 1.

Let us now construct the physical vacuum and consider one-string excitation on it. First consider the physical vacuum [8]. This is obtained by filling the Dirac sea. The distribution of the Dirac sea modes in the rapidity space can be determined by imposing periodic boundary conditions (PBC) on the Dirac sea mode:

$$\chi(x_i = 0) = \chi(x_i = L) \qquad i = 1, 2, \ldots, N \tag{2.14}$$

where L is the system length. From (2.2) and (2.14) we have

$$m_0 \sinh \alpha_i = \frac{2\pi n_i}{2L} + \frac{1}{L} \sum_j \phi(\alpha_i - \alpha_j) \tag{2.15}$$

where n_i are integers. In the limit $L \to \infty$, we define the density distribution of Dirac sea modes, $\rho(\alpha)$, by

$$\rho(\alpha) = \lim_{L \to \infty} \frac{1}{L(\alpha_{i+1} - \alpha_i)}. \tag{2.16}$$

It is noted that, by continuity from the free fermion theory,

$$n_{i+1} = n_i + 1 \tag{2.17}$$

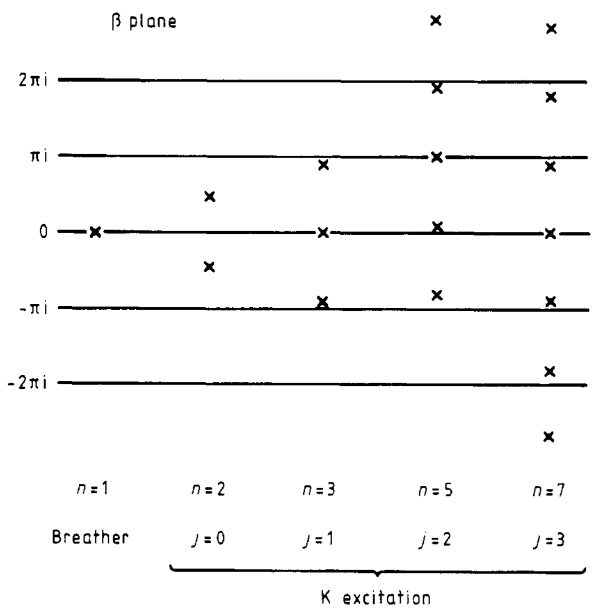

Figure 1. Allowed strings for $P_0 = 2 + \frac{1}{3}$.

in the physical vacuum. Subtracting (2.15) from (2.15) with i replaced by $i+1$, and using (2.16) and (2.17) gives

$$m_0 \cosh \alpha = 2\pi\rho(\alpha) + \int_{-\Lambda}^{\Lambda} \mathrm{d}\alpha' \frac{\partial}{\partial\alpha} \phi(\alpha-\alpha')\rho(\alpha') \tag{2.18}$$

where Λ is a rapidity cutoff associated with mass renormalisation. The integral equation (2.18) can be solved via Fourier transformation. The solution is

$$\rho(\alpha) = \frac{1}{4\mu} M_s \cosh \gamma\alpha \tag{2.19}$$

where $\gamma = \pi/2\mu$ and

$$M_s = \frac{m_0 \tan(\pi\gamma)}{\pi(\gamma-1)} \exp[(1-\gamma)\Lambda]. \tag{2.20}$$

It will become clear that M_s represents the physical mass of the SG soliton at zero temperature.

Next, consider one n-string excitation upon the physical vacuum [8, 9, 13]. In the zero-charge sector, to which we restrict ourselves in this paper, the n-string should accompany n holes in the Dirac sea. There are two types of holes. Some holes are tied to the string, i.e. the real part of their rapidities is equal to that of the string (α_n). Other holes are free from the string, i.e. the real parts of their rapidities can be any values other than α_n, thus representing additional freedoms. We shall refer to the two types of holes as 'fixed' and 'free' holes. The n-string, along with the holes tied to it, will be treated as an elementary excitation and each of the free holes regarded as an elementary excitation. The number of holes tied to an n-string can be found by examining the discontinuity at $\alpha = \alpha_n$ of the phase shift, $\Phi(\alpha, \alpha_n)$, which the n-string experiences upon the collision with a Dirac sea mode at $\alpha + \mathrm{i}\pi$. Using (2.8) and (2.9), we have

$$\begin{aligned}\Phi(\alpha, \alpha_n) &= \sum_{l_n=-(n-1)}^{n-1} \phi(\alpha + \mathrm{i}\pi - \beta_{l_n}) \\ &= G_{n+1}(\alpha, \alpha_n) + G_{n-1}(\alpha, \alpha_n) - A(n)\pi\varepsilon(\alpha-\alpha_n)\end{aligned} \tag{2.21}$$

where

$$G_m(\alpha, \alpha_n) = 2\tan^{-1}\{\tanh \tfrac{1}{2}(\alpha-\alpha_n) \cot \tfrac{1}{2}[m\mu - (n-B)\pi]\} \tag{2.22}$$

and

$$A(n) = n + 2(I_{n+1} + I_{n-1}) + 2(1-B) \tag{2.23}$$

where the integer I_m is defined by (using the Gauss symbol [ ])

$$I_m = [m\mu/2\pi - (n-B)/2]. \tag{2.24}$$

For allowed n as given by (2.13), we can show that

$$I_{n+1} = I_{n-1}. \tag{2.25}$$

Now to see that the quantity $A(n)$ represents the number of fixed holes, we consider PBC for Dirac sea modes $\bar{\beta}_i = \bar{\alpha}_i + \mathrm{i}\pi$ in the presence of the n-string

$$m_0 \sinh \bar{\alpha}_i = \frac{2\pi \bar{n}_i}{L} + \frac{1}{L}\sum_j \phi(\bar{\alpha}_i - \bar{\alpha}_j) + \frac{1}{L}\Phi_n(\bar{\alpha}_i, \alpha_n). \tag{2.26}$$

Comparing (2.26) for two neighbouring Dirac sea modes $\bar{\alpha}_m$ and $\bar{\alpha}_{m+1}$ such that $\bar{\alpha}_m < \alpha_n < \bar{\alpha}_{m+1}$, we have

$$\frac{2\pi\bar{n}_{m+1}}{L}+\frac{1}{L}\Phi_n(\bar{\alpha}_{m+1},\alpha_n)=\frac{2\pi\bar{n}_m}{L}+\frac{1}{L}\Phi_n(\bar{\alpha}_m,\alpha_n)+\frac{2\pi}{L} \tag{2.27}$$

or

$$\bar{n}_{m+1}-\bar{n}_m=A(n)+1. \tag{2.28}$$

Equation (2.28) means that $A(n)$ Dirac sea modes disappear from the physical vacuum, or equivalently, $A(n)$ holes are created at α_n due to the presence of the n-string. Figure 2 illustrates the three-string excitation from the physical vacuum for the case $P_0=2+\frac{1}{3}$. Note that for $n=3$, $B=0$ and $\mu=4\pi/7$, (2.23)-(2.25) give $A(3)=1$ and therefore we have one fixed hole.

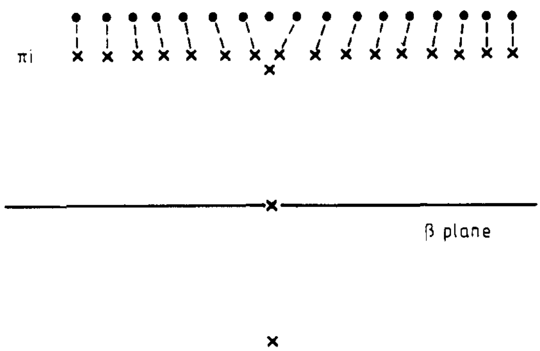

Figure 2. The three-string excitation from the physical vacuum for the case $P_0=2+\frac{1}{3}$. Full circles represent unperturbed Dirac sea modes, whereas crosses at Im $\beta=\pi$ perturbed ones.

To calculate the physical energy and physical momentum of elementary excitations, we have to determine the vacuum polarisation in the presence of the n-string. Subtracting (2.15) from (2.26), and defining

$$F(\alpha)=\lim_{L\to\infty}(\bar{\alpha}_i-\alpha_i)/(\alpha_{i+1}-\alpha_i) \tag{2.29}$$

we obtain

$$2\pi F(\alpha)=\Phi_n(\alpha,\alpha_n)-A(n)\phi(\alpha-\alpha_n)-\int_{-\infty}^{\infty}\mathrm{d}\alpha'\,F(\alpha')\frac{\partial}{\partial\alpha}\phi(\alpha-\alpha') \tag{2.30}$$

where we have used (2.18) and put $\Lambda=\infty$ because the integration here converges. Equation (2.30) has the same structure as (2.18) and can be solved by the Fourier transform method. Now with (2.4), the physical energy of the n-string can be computed by subtracting vacuum-state energy from the excited-state energy:

$$E_n(\alpha_n)=\sum_{l_n} m_0\cosh\beta_{l_n}+A(n)m_0\cosh\alpha_n-m_0\int_{-\Lambda}^{\Lambda}\mathrm{d}\alpha\,F(\alpha)\sinh\alpha. \tag{2.31}$$

Substituting the Fourier transform of $F(\alpha)$ obtained from (2.30) into (2.31) gives the physical energy of the n-string as

$$E_n(\alpha)=\begin{cases}2M_s\sin[\frac{1}{2}n\pi(2\gamma-1)]\cosh\gamma\alpha & n=1,2,\ldots,\nu_1-1\\ 0 & \text{otherwise.}\end{cases} \tag{2.32a}$$

Similarly, the physical momentum is given as

$$P_n(\alpha)=\frac{1}{\gamma}\frac{\mathrm{d}}{\mathrm{d}\alpha}E_n(\alpha). \tag{2.32b}$$

The first spectrum of (2.32) ($n=1,\ldots,\nu_1-1$) is precisely the SG breather spectrum of Dashen, Hasslacher and Neveu (DHN) [14]. The strings of vanishing physical energy are ($n=\nu_1$ and $1+j\nu_1$; $j=1,\ldots,\nu_2$) called the Korepin excitations (K excitations). Finally the physical energy of a free hole is similarly calculated by replacing the first two terms in (2.30) and (2.31) by $-\phi(\alpha-\alpha_{\mathrm{h}})$ and $m_0\cosh\alpha_{\mathrm{h}}$, respectively. We have

$$E_{\mathrm{hole}}(\alpha_{\mathrm{h}})=M_{\mathrm{s}}\cosh\gamma\alpha_{\mathrm{h}}. \tag{2.33}$$

The physical momentum of a hole is given by the same formula as (2.32*b*). It is noted that since a hole excitation in the MTM corresponds to an antisoliton excitation in the SG theory, M_{s} should be identified as the antisoliton (or soliton) mass, whose coupling constant dependence was studied by DHN [14]:

$$M_{\mathrm{s}}=M_{\mathrm{s}}^0\mu/\omega \tag{2.34}$$

where M_{s}^0 is the free soliton mass. We put $M_{\mathrm{s}}^0=1$ hereafter.

3. Multiple elementary excitations and two-body S matrices

In the previous section, we studied one elementary excitation in the physical vacuum. Our next step is to consider simultaneous excitations of many breathers, holes and K excitations and quantise their energies by imposing PBC. This leads to the determination of renormalised two-body S matrices between elementary excitations. Korepin first calculated two-body S matrices of the MTM [9], but his procedures are not quite clear and the S matrices obtained contain undetermined constants. A natural and unambiguous way of determining the S matrices is to impose PBC on elementary excitations as well as the Dirac sea modes.

In the presence of multiple elementary excitations equation (2.30) for the vacuum polarisation $F(\alpha)$ as defined by (2.29) is now replaced by

$$2\pi F(\alpha)=\sum_n[\Phi_n(\alpha,\alpha_n)-A(n)\phi(\alpha-\alpha_n)]-\sum_k\phi(\alpha-\alpha_k)$$
$$-\int_{-\infty}^{\infty}\mathrm{d}\alpha'\,F(\alpha')\frac{\partial}{\partial\alpha}\phi(\alpha-\alpha') \tag{3.1}$$

where Σ_n means a summation over n-strings and Σ_k over free holes. We note that the vacuum polarisation is an additive quantity, i.e.

$$F(\alpha)=\sum_n F_n(\alpha)+\sum_k F_k(\alpha) \tag{3.2}$$

where $F_n(\alpha)$ and $F_k(\alpha)$, respectively, satisfy (3.1) with the first two terms on the right replaced by $\Phi_n(\alpha,\alpha_n)-A(n)\phi(\alpha-\alpha_n)$ and $-\phi(\alpha-\alpha_k)$.

Now imposing PBC on the n-string gives

$$m_0\sum_{l_n}\sinh\beta_{l_n}=\frac{2\pi}{L}\times\text{integer}+\frac{1}{L}\sum_i{}'\,\Phi_n(\tilde\alpha_i,\alpha_n)+\frac{1}{L}\sum_{m\neq n}\sum_{l_m}\sum_{l_n}\phi(\beta_{l_m}-\beta_{l_n}) \tag{3.3}$$

where Σ_i' is over the occupied Dirac sea modes $\tilde{\beta}_i = \tilde{\alpha}_i + \mathrm{i}\pi$. Expanding the function $\Phi_n(\tilde{\alpha}_i, \alpha_n)$ at the unperturbed position α_i in the Dirac sea and summing over i gives

$$\sum_i{}' \Phi_n(\tilde{\alpha}_i, \alpha_n) = \sum_i \Phi_n(\alpha_i, \alpha_n) - \sum_k \Phi_n(\alpha_k, \alpha_n) - \sum_m A(m)\Phi_n(\alpha_m, \alpha_n) + \int \mathrm{d}\alpha\, F(\alpha)\frac{\partial}{\partial\alpha}\Phi_n(\alpha, \alpha_n) \tag{3.4}$$

where Σ_i is over all Dirac sea modes. A similar equation can be derived when only one n-string exists. Subtracting the resulting equation from (3.4) yields

$$\sum_i \Phi_n(\tilde{\alpha}_i, \alpha_n) = \sum_i \Phi_n(\bar{\alpha}_i, \alpha_n) - \sum_k \Phi_n(\alpha_k, \alpha_n) - \sum_{m\neq n} A(m)\Phi_n(\alpha_m, \alpha_n) + \sum_{m\neq n}\int_{-\infty}^{\infty} \mathrm{d}\alpha\, F_m(\alpha)\frac{\partial}{\partial\alpha}\Phi_n(\alpha, \alpha_n) \tag{3.5}$$

where $\bar{\alpha}_i$ represent real parts of the Dirac sea modes in the presence of the n-string alone.

On the other hand, subtracting (2.15) from (2.26) and summing over i gives

$$\frac{1}{L}\sum_i \Phi_n(\bar{\alpha}_i, \alpha_n) = \frac{1}{L}A(n)\sum_i \phi(\alpha_i - \alpha_n) + m_0\int_{-\Lambda}^{\Lambda} \mathrm{d}\alpha\, F_n(\alpha)\cosh\alpha. \tag{3.6}$$

Expanding $\phi(\alpha_i - \alpha_n)$ at the unperturbed position α_n^0 in the Dirac sea and summing over i gives

$$\frac{1}{L}\sum_i \phi(\alpha_i - \alpha_n) = \frac{1}{L}\sum_i \phi(\alpha_i - \alpha_n^0) - \int_{-\Lambda}^{\Lambda} \mathrm{d}\alpha\frac{\partial}{\partial\alpha}\phi(\alpha - \alpha_n^0)\rho(\alpha)(\alpha_n - \alpha_n^0). \tag{3.7}$$

Substituting (2.15) and (2.18) into (3.7) yields

$$\frac{1}{L}\sum_i \phi(\alpha_i - \alpha_n) = \frac{2\pi}{L}\times\text{integer} + \frac{2\pi}{L}F(\alpha_n) - m_0\sinh\alpha_n. \tag{3.8}$$

Using (3.6) and (3.8), we obtain the physical momentum of the n-string as

$$P_n(\alpha_n) = m_0\sum_{l_n}\sinh\beta_{l_n} - \frac{1}{L}\sum_i \Phi_n(\bar{\alpha}_i, \alpha_n) + A(n)\left(\frac{2\pi}{L}\times\text{integer} + \frac{2\pi}{L}A(n)F(\alpha_n)\right). \tag{3.9}$$

Substituting (3.3) and (3.5) into (3.9) gives

$$P_n(\alpha_n) = \frac{2\pi}{L}\times\text{integer} + \frac{1}{L}\sum_{m\neq n}\Delta_{nm} + \frac{1}{L}\sum_k \Delta_{nk} \tag{3.10a}$$

where

$$\Delta_{nm} = \sum_{l_m}\sum_{l_n}\phi(\beta_{l_m} - \beta_{l_n}) - A(m)\Phi_n(\alpha_m, \alpha_n) + 2\pi A(n)F_m(\alpha_n) + \int_{-\infty}^{\infty} \mathrm{d}\alpha\, F_m(\alpha)\frac{\partial}{\partial\alpha}\Phi_n(\alpha, \alpha_n) \tag{3.10b}$$

and

$$\Delta_{nk} = -\phi_n(\alpha_k, \alpha_n) + 2\pi A(n)F_k(\alpha_n) + \int_{-\infty}^{\infty} \mathrm{d}\alpha\, F_k(\alpha)\frac{\partial}{\partial\alpha}\Phi_n(\alpha, \alpha_n). \tag{3.10c}$$

As for the free hole, we note that its position α_k is shifted from the corresponding position α_k^0 in the unperturbed Dirac sea due to the presence of other holes and strings. Expanding the physical momentum of the kth hole at α_k^0 and using (2.19) and (2.29), we have

$$\begin{aligned} P_{\mathrm{n}}(\alpha_k) &= M_{\mathrm{s}} \sinh \gamma\alpha_k = M_{\mathrm{s}} \sinh \gamma\alpha_k^0 + M_{\mathrm{s}}\gamma \cosh \alpha_k(\alpha_k - \alpha_k^0) \\ &= \frac{2\pi}{L} \times \text{integer} + \frac{2\pi}{L} F(\alpha_k). \end{aligned} \tag{3.11}$$

With (3.2) we can write (3.11) as

$$P_{\mathrm{h}}(\alpha_k) = \frac{2\pi}{L} \times \text{integer} + \frac{1}{L} \sum_{j \neq k} \Delta_{kj} \tag{3.12}$$

where j runs over strings and holes and

$$\Delta_{kj} = 2\pi F_j(\alpha_k). \tag{3.13}$$

Notice that the Δ as defined by (3.10*b*), (3.10*c*) and (3.13) are composed of bare phase shifts between elementary excitations and additional phase shifts due to the vacuum polarisation in the presence of other elementary excitations. Thus the Δ represent renormalised phase shifts for the scattering between two elementary excitations. Δ can be evaluated again by the Fourier transform method. An example of calculations is given in the appendix. The results are

$\Delta_{mn}^{\mathrm{bb}}(\alpha)$($m$th breather–$n$th breather)

$$= \xi(\alpha, m+n, 0) + \xi(\alpha, |m-n|, 0) + 2 \sum_{l=1}^{\min(m,n)-1} \xi(\alpha, m+n-2l, 0) \tag{3.14}$$

$$\Delta_n^{\mathrm{bh}}(\alpha) \ (n\text{th breather–hole}) = \sum_{l=1}^{n} \xi(\alpha, n-2l+1, 1) \tag{3.15}$$

where $m, n = 1, 2, \ldots, \nu_1 - 1$ and

$$\xi(\alpha, k, D) = -\mathrm{i} \ln\left(\frac{\sinh \gamma\alpha - \mathrm{i} \sin \gamma(k\omega + \pi D)}{\sinh \gamma\alpha + \mathrm{i} \sin \gamma(k\omega + \pi D)}\right) \tag{3.16}$$

$$\Delta^{\mathrm{hh}}(\alpha)(\text{hole–hole}) = \int_{-\infty}^{\infty} \mathrm{d}y \frac{\sin \alpha y}{y} \frac{\sinh(\pi - 2\mu)y}{\sinh \pi y + \sinh(\pi - 2\mu)y}. \tag{3.17}$$

We find that breathers do not interact with K excitations. We also find that the $n = 1 + \nu_1\nu_2$ K excitation does not interact with any other elementary excitations. Therefore, the only possible way for this excitation to affect the MTM thermodynamics is through the charge-neutrality constraint, because this excitation carries a non-zero charge. In the variational formulation of the thermodynamics below, this charge-neutrality condition can be treated by introducing a chemical potential conjugate to the charge. However, since antisolitons have charges opposite to those of solitons, this chemical potential should be zero in order that, in the zero-charge sector, the theory is symmetric with respect to solitons and antisolitons. This means that the $n = 1 + \nu_1\nu_2$ K excitation does not affect the MTM thermodynamics. We thus discard this K excitation. For simplicity, we shall call the $n = \nu_1$ and $n = 1 + j\nu_1$ ($j = 1, 2, \ldots, \nu_2 - 1$) K excitations the zeroth and jth K excitations and denote them by $n = n_j$; $j = 0, 1, \ldots, \nu_2 - 1$ (see figure 1). For $i, j = 0, 1, 2, \ldots, \nu_2 - 1$, we have

$$\Delta_j^{\mathrm{Kh}}(\alpha)(j\text{th K excitation–hole}) = \zeta(\alpha, n_j - 1, M_j) \tag{3.18}$$

$-\Delta_{ij}^{KK}(\alpha)$($i$th K excitation-$j$th K excitation)

$$= \zeta(\alpha, n_i+n_j, M_i+M_j) + \zeta(\alpha, n_i+n_j, |M_i - M_j|) + 2\sum_{l=1}^{\min(M_i,M_j)-1} \zeta(\alpha, n_i+n_j, |M_i - M_j| + 2l) \tag{3.19}$$

where $M_j = [n_j - A(n_j)]/2$ and

$$\zeta(\alpha, n, M) = -\mathrm{i}\ln\left(\frac{\sinh[\pi(\alpha + \mathrm{i}\omega n - \mathrm{i}\pi M)/2\omega]}{\sinh[\pi(\alpha - \mathrm{i}\omega n + \mathrm{i}\pi M)/2\omega]}\right). \tag{3.20}$$

With (2.23)-(2.25) we can easily show that

$$M_0 = 1 \qquad \text{and} \qquad M_i = i \qquad \text{for } i = 1, 2, \ldots, \nu_2 - 1. \tag{3.21}$$

We thus complete the quantisation of physical momenta of elementary excitations. For later convenience, we put together (3.10) and (3.12) in a compact form. Let $\rho_i(\alpha)$ denote the density distribution of the i-kind excitation in the rapidity space. Then (3.10) and (3.12) can be written as

$$P_j(\alpha) = \frac{2\pi}{L} \times \text{integer} + \frac{1}{L}\sum_i \Delta_{ji} * \rho_i \tag{3.22}$$

where i and j run over breathers, free hole and K excitations and we have introduced a convenient notation

$$a * b = \int_{-\infty}^{\infty} \mathrm{d}\alpha'\, a(\alpha' - \alpha) b(\alpha'). \tag{3.23}$$

4. Thermodynamics and finite-temperature excitations

In § 3, we have quantised generic excited states and reached the renormalised phase shift between renormalised elementary excitations. In this section, we will perform remaining procedures in the BA variational formulation of the MTM thermodynamics.

Let us first introduce distribution densities of unoccupied α, $\tilde{\rho}_i(\alpha)$, for breathers, holes and K excitations. Then, differentiating (3.22) with respect to α gives

$$\frac{\mathrm{d}P_j(\alpha)}{\mathrm{d}\alpha} = 2\pi\,\mathrm{sgn}(j)[\rho_j(\alpha) + \tilde{\rho}_j(\alpha)] + \frac{1}{L}\sum_i \frac{\partial}{\partial\alpha}\Delta_{ji} * \rho_i \tag{4.1}$$

where $\mathrm{sgn}(j) = 1$ for j denoting breathers, free hole and the zeroth K excitation ($n = \nu_1$ string) and -1 for j denoting other K excitations. In a previous letter [12], we have argued that the negative $\mathrm{sgn}(j)$ for K excitations with $j = 1, 2, \ldots, \nu_2 - 1$ comes from the fact that the bare energies are negative for these K excitations. However, this argument is not satisfactory, because the physical momenta are zero for these K excitations. Therefore, we have determined $\mathrm{sgn}(j)$ for K excitations by explicitly looking at the signs of $\partial\Delta_{ji}/\partial\alpha$ in (4.1). Equation (4.1) provides a relationship between densities $\rho_j(\alpha)$ and $\tilde{\rho}_j(\alpha)$ since $\mathrm{d}P_j(\alpha)/\mathrm{d}\alpha$ are known.

We now derive the other relationship between ρ_j and $\tilde{\rho}_j$. The internal energy, E, is

$$E = L\sum_i \int_{-\infty}^{\infty} \mathrm{d}\alpha\, E_i(\alpha)\rho_i(\alpha) \tag{4.2}$$

where i runs over breathers and free holes. The local entropy, ΔS, in the interval $(\alpha, \alpha+\Delta\alpha)$ is given by

$$\Delta S = \sum_i \ln\left(\frac{(L\rho_i\Delta + L\tilde{\rho}_i\Delta)!}{(L\rho_i\Delta)!(L\tilde{\rho}_i\Delta)!}\right) \tag{4.3}$$

where we have put Boltzmann's constant $=1$. Therefore, the total entropy, S, is

$$S = L\sum_i \int_{-\infty}^{\infty} \mathrm{d}\alpha[(\rho_i+\tilde{\rho}_i)\ln(\rho_i+\tilde{\rho}_i) - \rho_i \ln\rho_i - \tilde{\rho}_i \ln\tilde{\rho}_i]. \tag{4.4}$$

Thus the free energy $F = E - TS$ can be written in terms of ρ_i and $\tilde{\rho}_i$. Using (4.1), we can finally write the free energy in terms of ρ_i alone. In minimising the free energy with respect to variations of densities $\delta\rho_i$, we keep it in mind that the constant-charge restriction is automatically satisfied in the neutral-charge sector, because the associated chemical potential should be zero (see § 3). The minimisation of the free energy gives

$$\varepsilon_j(\alpha) = E_j(\alpha) + \frac{T}{2\pi}\sum_i \mathrm{sgn}(i)\frac{\partial}{\partial\alpha}\Delta_{ij} * \ln[1+\exp(-\varepsilon_i/T)] \tag{4.5}$$

where the temperature, T, is measured in the unit of the zero-temperature free soliton mass, M_s^0, and we have defined

$$\tilde{\rho}_j/\rho_j = \exp(\varepsilon_j/T). \tag{4.6}$$

Equations (4.1) and (4.5) are the basic equations which describe the MTM-SG thermodynamics for $P_0 = \nu_1 + 1/\nu_2$ and ν_1 and $\nu_2 \geqslant 2$. The quantities $\varepsilon_j(\alpha)$ are the MTM correspondents of the Yang and Yang ε function in the non-linear Schrödinger model and play a central role in the MTM thermodynamics.

Two remarks are in order at this point. First, we can obtain the free energy in terms of ε_j by carrying out

$$\sum_j \int_{-\infty}^{\infty} \mathrm{d}\alpha\left(\text{equation (4.1) } \mathrm{sgn}(j)\frac{T}{2\pi}\ln[1+\exp(-\varepsilon_j/T)] - \text{equation (4.5)}\rho_j\right)$$

$$F/L = \sum_j \int_{-\infty}^{\infty} \mathrm{d}\alpha\{E_j\rho_j - \varepsilon_j\rho_j - T\rho_j[1+\exp(\varepsilon_j/T)]\ln[1-\exp(-\varepsilon_j/T)]\}$$

$$= -\frac{T\gamma}{2\pi}\sum_j \int_{-\infty}^{\infty} \mathrm{d}\alpha\, E_j(\alpha)\ln[1+\exp(-\varepsilon_j(\alpha)/T)]. \tag{4.7}$$

Note that only holes and breathers contribute to the free energy, since the physical energies are zero for all the K excitations. This and other unphysical aspects of the K excitations have been previously pointed out [13]. However, the existence of these exotic excitations is crucial for us to be able to incorporate the soliton-antisoliton backscattering in the Bethe wavefunction which is written in terms of forward scatterings only. Second, we can show that $\varepsilon_j(\alpha)$ represents the excitation spectrum at finite temperatures. (Here j denotes only holes and breathers due to the above argument.) To see this, consider an excitation in which a j-kind elementary excitation at α_k is moved to $\bar{\alpha}_k$. This excitation is accompanied by a polarisation of other modes $\alpha_{i,l} \to \bar{\alpha}_{i,l}$, where i runs over all kinds of excitations and l labels their rapidities. Therefore, the excitation energy, ΔE, is given by

$$\Delta E = E_j(\alpha_k) - E_j(\bar{\alpha}_k) + \sum_i \int_{-\infty}^{\infty} \mathrm{d}\alpha\, \rho_i(\alpha)\chi_i(\alpha)\frac{\mathrm{d}E_i(\alpha)}{\mathrm{d}\alpha} \tag{4.8}$$

where $\chi_i(\alpha)$ describes the polarisation of the i-kind elementary excitations:

$$\chi_i(\alpha)=\lim_{L\to\infty} L(\bar{\alpha}_{i,l}-\alpha_{i,l}). \tag{4.9}$$

The equations which determine $\chi_i(\alpha)$ can be obtained by subtracting PBC for the distribution $\{\alpha_{i,l}\}$ from those for $\{\bar{\alpha}_{i,l}\}$ (cf (3.22)) as

$$\frac{\mathrm{d}P_j(\alpha)}{\mathrm{d}\alpha}\chi_i(\alpha)=\Delta_{ij}(\bar{\alpha}_k-\alpha)-\Delta_{ij}(\alpha_k-\alpha)$$

$$+\sum_{j'}\int_{-\infty}^{\infty}\mathrm{d}\alpha'\frac{\partial}{\partial\alpha}\Delta_{ij'}(\alpha'-\alpha)[\chi_i(\alpha)-\chi_{j'}(\alpha')]\rho_{j'}(\alpha'). \tag{4.10}$$

Substituting (4.1) into (4.10) gives

$$2\pi\,\mathrm{sgn}(i)\chi_i\rho_i[1+\exp(\varepsilon_i/T)]=\Delta_{ij}(\bar{\alpha}_k-\alpha)-\Delta_{ij}(\alpha_k-\alpha)-\sum_{j'}\frac{\partial}{\partial\alpha}\Delta_{ij}*(\rho_{j'}\chi_{j'}). \tag{4.11}$$

We rewrite (4.5) (through integration by parts) as

$$\varepsilon_{j'}(\alpha)-E_{j'}(\alpha)=\frac{1}{2\pi}\sum_i \mathrm{sgn}(i)\int_{-\infty}^{\infty}\mathrm{d}\alpha'\,\Delta_{ij'}(\alpha'-\alpha)[1+\exp(\varepsilon_i(\alpha')/T)]^{-1}\frac{\mathrm{d}\varepsilon_i(\alpha')}{\mathrm{d}\alpha'}. \tag{4.12}$$

Carrying out

$$\sum_{j'}\int_{-\infty}^{\infty}\mathrm{d}\alpha\,\rho_{j'}\chi_{j'}\frac{\partial}{\partial\alpha}(\text{equation (4.12)})$$

and using (4.11), we obtain

$$\sum_i\int_{-\infty}^{\infty}\mathrm{d}\alpha\,\rho_i\chi_i\,\mathrm{d}E_i(\alpha)/\mathrm{d}\alpha=\varepsilon_j(\bar{\alpha}_k)-E_j(\bar{\alpha}_k)-\varepsilon_j(\alpha_k)+E_j(\alpha_k). \tag{4.13}$$

From (4.8) and (4.12) we finally obtain the excitation energy

$$\Delta E=\varepsilon_j(\bar{\alpha}_k)-\varepsilon_j(\alpha_k). \tag{4.14}$$

The above argument can be generalised to the case of finite number of elementary excitations of various kinds. Thus, we can interpret $\varepsilon_j(\alpha)$ as the thermally renormalised energy of the j-kind elementary excitation.

Here it is noted that in the zero-charge sector, densities of solitons and antisolitons are by symmetry the same at any point in the rapidity space. Since K excitations carry only charges and not free energies, the contribution of holes to the local (in the rapidity space) free energy should be twice as much as that of solitons. In light of (4.7), we have claimed in a previous letter [12] that

$$1+\exp(-\varepsilon^{\mathrm{h}}/T)=[1+\exp(-\varepsilon^{\mathrm{s}}/T)]^2 \tag{4.15}$$

with ε^{s} representing the soliton energy. However, (4.15) is not conclusive because there is no guarantee that the contribution of solitons to the local free energy can also be written as

$$-\frac{T\gamma}{2\pi}E^{\mathrm{h}}(\alpha)\ln[1+\exp(-\varepsilon^{\mathrm{s}}(\alpha)/T)]. \tag{4.16}$$

Fortunately, however, our conjecture in the above has recently been given a justification in the factorised S-matrix formulation of the MTM thermodynamics [15]. Thus, we can safely use (4.15) below, although it cannot be shown within a framework of the BA formalism.

Returning to (4.5), we note that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\Delta_{j0}=-\frac{\mathrm{d}}{\mathrm{d}\alpha}\Delta_{j,\nu_2-1}$$

for j denoting hole and K excitations, and 0 and ν_2-1, respectively, denoting the zeroth and (ν_2-1)th K excitations. Therefore

$$\varepsilon_0^{\mathrm{K}}(\alpha)=-\varepsilon_{\nu_2-1}^{\mathrm{K}}(\alpha). \tag{4.17}$$

This demonstrates the unphysical aspect of K excitations, because either one of the two 'thermally renormalised energies' should be negative. A simple substitution of (4.17) into (4.5) contains a term proportional to $\ln(1+\exp(\varepsilon_{\nu_2-1}^{\mathrm{K}}/T))$ which diverges in the limit $\nu_2\to\infty$ (see (4.25) below). To avoid this difficulty we first solve the equation for $\eta_{\nu_2-1}^{\mathrm{K}}\equiv\varepsilon_{\nu_2-1}^{\mathrm{K}}/T$, and then substitute the resulting $\eta_{\nu_2-1}^{\mathrm{K}}$ into the remaining equations. Although calculations are rather lengthy, the procedures are straightforward. First, we carry out the phase shift derivatives to obtain

$$\begin{aligned}\eta_{\nu_2-1}^{\mathrm{K}}-S_2^-*\eta_{\nu_2-1}^{\mathrm{K}}&\\&=\sum_{i=1}^{\nu_2-2}(S_{i+1}^++S_{i-1}^+)*\ln[1+\exp(-\eta_i^{\mathrm{K}})]+2S_2^-*\ln[1+\exp(-\eta_{\nu_2-1}^{\mathrm{K}})]\\&\quad+S_1^+*\ln[1+\exp(-\eta_{\mathrm{h}})]\end{aligned} \tag{4.18}$$

where we have defined

$$S_j^{\mp}(\alpha)=\frac{P_0\sin(\pi P_0 j)}{\cosh P_0\alpha\pm\cos(\pi P_0 j)}. \tag{4.19}$$

Equation (4.18) can be solved by the Fourier transform method to give $\eta_{\nu_2-1}^{\mathrm{K}}$ in terms of the quantities $\ln(1+\mathrm{e}^{-\eta})$. Next, substituting $\eta_{\nu_2-1}^{\mathrm{K}}$ into terms in (4.5) containing

$$\frac{\partial}{\partial\alpha}\Delta_{\nu_2-1,i}*\eta_{\nu_2-1}^{\mathrm{K}}$$

where i denotes hole or K excitations, and integrating once, we obtain η^{h} and $\eta_j^{\mathrm{K}}(j=1,2,\ldots,\nu_2-2)$ in terms of the quantities $\ln(1+\mathrm{e}^{-\eta})$. The result is

$$\eta_n^{\mathrm{b}}=\frac{E_n^{\mathrm{b}}}{T}+\sum_{m=1}^{\nu_1-1}\theta_{mn}^{\mathrm{bb}}*\ln[1+\exp(-\eta_m^{\mathrm{b}})]+\theta_n^{\mathrm{hb}}*\ln[1+\exp(-\eta^{\mathrm{h}})]$$

$$n=1,2,\ldots,\nu_1-1 \tag{4.20a}$$

$$\begin{aligned}\eta^{\mathrm{h}}=\frac{E^{\mathrm{h}}}{T}&+\sum_{n=1}^{\nu_1-1}\theta_n^{\mathrm{bh}}*\ln[1+\exp(-\eta_n^{\mathrm{b}})]+(\theta_1^{\mathrm{hh}}+\theta_2^{\mathrm{hh}})*\ln[1+\exp(-\eta^{\mathrm{h}})]\\&+\sum_{j=1}^{\nu_2-1}\theta_j^{\mathrm{Kh}}*\ln[1+\exp(-\eta_j^{\mathrm{K}})]\end{aligned} \tag{4.20b}$$

$$\eta_j^{\mathrm{K}}=\frac{1}{1+\delta_{j,\,\nu_2-1}}\left(\theta_j^{\mathrm{hK}}*\ln[1+\exp(-\eta^{\mathrm{h}})]+\sum_{i=1}^{\nu_2-1}\theta_{ij}^{\mathrm{KK}}*\ln[1+\exp(-\eta_i^{\mathrm{K}})]\right)$$

$$j=1,2,\ldots,\nu_2-1 \qquad (4.20c)$$

where

$$\theta_{mn}^{\mathrm{bb}}=\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\alpha}\Delta_{mn}^{\mathrm{bb}} \qquad (4.21a)$$

$$\theta_n^{\mathrm{hb}}=\theta_n^{\mathrm{bh}}=\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\alpha}\Delta_n^{\mathrm{bh}} \qquad (4.21b)$$

$$\theta_1^{\mathrm{hh}}=\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\alpha}\Delta^{\mathrm{hh}} \qquad (4.21c)$$

and

$$\theta_{ij}^{\mathrm{KK}}=S(\alpha,i+j)+S(\alpha,|i-j|)+2\sum_{l=1}^{\min(i,j)-1}S(\alpha,|i-j|+2l) \qquad (4.22a)$$

$$\theta_j^{\mathrm{hK}}=-\theta_j^{\mathrm{Kh}}=S(\alpha,j) \qquad (4.22b)$$

$$\theta_2^{\mathrm{hh}}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}y\,\frac{\cos(\alpha y)\sinh(\omega y/\nu_2)}{\sinh[\omega y(2-1/\nu_2)]+\sinh(\omega y/\nu_2)} \qquad (4.22c)$$

where

$$S(\alpha,k)=\frac{1}{\omega(1-1/\nu_2)}\,\frac{\cosh\{P_0\nu_2\alpha/[2(\nu_2-1)]\}\sin\{k\pi/[2(\nu_2-1)]\}}{\cosh[P_0\nu_2\alpha/(\nu_2-1)]-\cos[k\pi/(\nu_2-1)]}. \qquad (4.23)$$

Solving the coupled integral equations (4.20)-(4.23) we can find the excitation spectra at finite temperatures, the free energy and hence all the thermodynamic quantities. For finite ν_1 and ν_2, the above basic equations are beyond analytical treatments, and we have to resort to numerical calculations. On the other hand, in the limit $\nu_2\to\infty$ we can partly solve the equations and simplify them as follows. In this limit, $S(\alpha,k)\to\delta(\alpha)$ in (4.23), and (4.20c) for finite j becomes

$$\eta_j^{\mathrm{K}}=\ln[1+\exp(-\eta^{\mathrm{h}})]+\sum_{i<j}2i\ln[1+\exp(-\eta_i^{\mathrm{K}})]$$
$$+(2j-1)\ln[1+\exp(-\eta_j^{\mathrm{K}})]+\sum_{i>j}^{\infty}2j\ln[1+\exp(-\eta_i^{\mathrm{K}})]. \qquad (4.24)$$

A similar equation to (4.24) was solved previously by Johnson [16]. Following similar procedures gives

$$[1+\exp(\eta_j^{\mathrm{K}})]^{1/2}=j+[1+\exp(-\eta^{\mathrm{h}})]^{1/2} \qquad \text{for } j\geqslant 1. \qquad (4.25)$$

With (4.24) for $j=1$ and (4.25), the fourth term on the right of (4.20b) becomes

$$-\sum_{j=1}^{\infty}\ln[1+\exp(-\eta_j^{\mathrm{K}})]=-\ln\{1+[1+\exp(-\eta^{\mathrm{h}})]^{-1/2}\}. \qquad (4.26)$$

Note also that $\theta_2^{\mathrm{hh}}\to 0$ when $\nu_2\to\infty$ as is seen from (4.22c). In this way, (4.20b) becomes

$$\eta^{\mathrm{h}}=\frac{E^{\mathrm{h}}}{T}+\sum_{n=1}^{\nu_1-1}\theta_n^{\mathrm{bh}}*\ln[1+\exp(-\eta_n^{\mathrm{b}})]$$
$$+\theta_1^{\mathrm{hh}}*\ln[1+\exp(-\eta^{\mathrm{h}})]-\ln\{1+[1+\exp(-\eta^{\mathrm{h}})]^{-1/2}\}. \qquad (4.20b')$$

(4.20a, b') provide closed equations for η^{h} and η^{b}_n ($n=1,2,\ldots,\nu_1-1$). These equations were also derived by Imada *et al* [2] based on the Takahashi-Suzuki formalism [3].

Now a puzzling question arises concerning the basic equations for the cases $P_0=\nu_1$ and ν_1+0. The two sets of equations differ from each other, although it was proved by Araki [17] that the free energy of the system is an analytic function of the coupling constant. The case $P_0=\nu_1$ was carefully examined by the present authors in a previous letter [13]. In this case we have $n=1,2,\ldots,\nu_1-2$ breathers, solitons and antisolitons as elementary excitations. By symmetry, solitons and antisolitons behave in precisely the same manner and the basic thermodynamic equations for the case $P_0=\nu_1$ are

$$\tilde{\eta}^{\mathrm{b}}_n=\frac{E^{\mathrm{b}}_n}{T}+\sum_{m=1}^{\nu_1-2}\theta^{\mathrm{bb}}_{mn}*\ln[1+\exp(-\tilde{\eta}^{\mathrm{b}}_m)]+2\theta^{\mathrm{bh}}_n*\ln[1-\exp(-\tilde{\eta}^{\mathrm{s}})]$$

$$n=1,2,\ldots,\nu_1-2 \qquad (4.27a)$$

$$\tilde{\eta}^{\mathrm{s}}=\frac{E^{\mathrm{h}}}{T}+\sum_{n=1}^{\nu_1-2}\theta^{\mathrm{bh}}_n*\ln[1+\exp(-\tilde{\eta}^{\mathrm{b}}_n)]+2\theta^{\mathrm{hh}}_1*\ln[1+\exp(-\tilde{\eta}^{\mathrm{s}})]. \qquad (4.27b)$$

The free energy is given in terms of $\tilde{\eta}$ as

$$F/L=-\frac{\gamma T}{2\pi}\sum_j\int_{-\infty}^{\infty}\mathrm{d}\alpha\, E_j(\alpha)\ln[1+\exp(-\tilde{\eta}_j)] \qquad (4.28)$$

where j runs over ν_1-2 kinds of breather, soliton and antisoliton. (4.27a, b) were also derived by Fowler [18a] based on [3]†. To see a smooth change of the free energy at the point $P_0=\nu_1$, we examine (4.20a) for $j=\nu_1-1$. From (3.14)-(3.17) and (4.21) we can show that

$$\theta^{\mathrm{bb}}_{\nu_1-1,\nu_1-1}=2\theta^{\mathrm{bh}}_{\nu_1-1}-\delta(\alpha) \qquad (4.29a)$$

$$\theta^{\mathrm{bb}}_{n,\nu_1-1}=2\theta^{\mathrm{bh}}_n \qquad \text{for } n\neq\nu_1-1 \qquad (4.29b)$$

$$\theta^{\mathrm{bh}}_{\nu_1-1}=2\theta^{\mathrm{hh}}_1+\delta(\alpha). \qquad (4.29c)$$

With these relationships, (4.20a) for $n=\nu_1-1$ becomes

$$\begin{aligned}\ln[1+\exp(\eta^{\mathrm{b}}_{\nu_1-1})]&-\ln[1+\exp(-\eta^{\mathrm{h}})]\\ &=2\left(\frac{E^{\mathrm{h}}}{T}+\sum_{m=1}^{\nu_1-1}\theta^{\mathrm{bh}}_m*\ln[1+\exp(-\eta^{\mathrm{b}}_m)]+\theta^{\mathrm{hh}}_1*\ln[1+\exp(-\eta^{\mathrm{h}})]\right)\\ &=2[\![\eta^{\mathrm{h}}+\ln\{1+[1+\exp(-\eta^{\mathrm{h}})]^{-1/2}\}]\!]\end{aligned} \qquad (4.30)$$

where the second equality is due to (4.20b'). Therefore

$$[1+\exp(\eta^{\mathrm{b}}_{\nu_1-1})]^{1/2}=\exp(\eta^{\mathrm{h}})\{1+[1+\exp(-\eta^{\mathrm{h}})]^{1/2}\}. \qquad (4.31)$$

Substituting $\eta^{\mathrm{b}}_{\nu_1-1}$ back into (4.20a, b') and identifying $\eta^{\mathrm{b}}_n=\tilde{\eta}^{\mathrm{b}}_n$ ($n=1,2,\ldots,\nu_1-2$) and (cf (4.15))

$$[1+\exp(-\eta^{\mathrm{s}})][1+\exp(-\eta^{\mathrm{b}}_{\nu_1-1})]=1+\exp(-\tilde{\eta}^{\mathrm{s}}) \qquad (4.32)$$

reproduces the basic equations for the case $P_0=\nu_1$, (4.27). Through the same procedure, equation (4.7) for the free energy becomes (4.28). It is thus proved that both the free

† An exact enumeration of the basic equation in [18] was followed by an analytic solution (see [18a]). For recent developments see [18b].

energy and breather masses change smoothly with the coupling constant P_0. However, at finite temperatures, it is clear that $\varepsilon^{s}(0) \neq \tilde{\varepsilon}^{s}(0)$, i.e. the soliton mass is discontinuous at the point $P_0 = \nu_1$. The origin of such discontinuity in the soliton mass is in the fact that the longest breather dissociates into a soliton-antisoliton pair when P_0 passes ν_1 from larger values. Since the free energy is continuous at $P_0 = \nu_1$, the soliton mass should decrease suddenly at this point to cover a sudden disappearance of parts of the free energy carried by the dissociated breathers. It is seen in figure 3 that the magnitude of discontinuity decreases with increasing coupling constant P_0 and increases with increasing temperature. A further discussion on the soliton-mass discontinuity will be given in the final section.

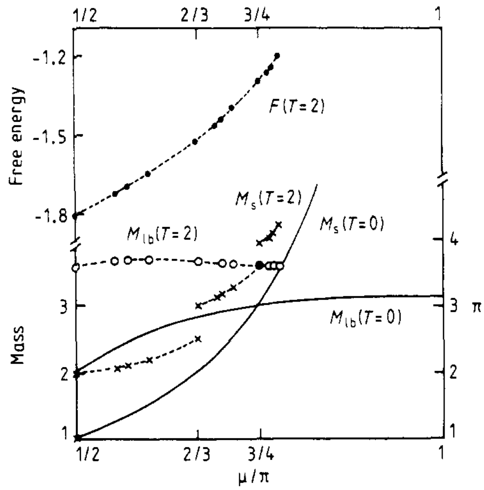

Figure 3. Plot of the free energy F (full circles), soliton mass (crosses) and lowest-breather mass $M_{\rm lb}$ (open circles) as functions of the coupling constant μ/π at $T = 2$. These discrete points are connected by broken curves for clarity. The soliton and lowest-breather masses at $T = 0$ are also plotted (full curves) for comparison. The mass and temperature are measured in units of the zero-temperature free soliton mass, M_{s}^{0}.

There are two cases in which we can solve the basic equations analytically: the free MTM limit $\mu = \pi/2 + 0$ $(P_0 = 2 + 0)$ and the free SG limit $\mu = \pi - 0$ $(P_0 = \nu_1 = \infty)$. By analysing the latter case, one of us [18] recently reached an essential understanding of the role of breathers in the quantum SG thermodynamics, which had been a challenging problem in soliton physics [6]. As for the former case, $\theta_1^{\rm hh} = 0$ and $\theta_1^{\rm bh} = \delta(\alpha)$ and (4.30) can easily be solved to give

$$[1+\exp(-\eta^{\rm h})]^{1/2} = \frac{Z+2}{Z+1} \qquad 1+\exp(-\eta_1^{\rm b}) = \frac{(Z+1)^2}{Z^2+2Z} \tag{4.33}$$

where $Z = \exp(E^{\rm h}/T)$. From (4.15) and (4.33) we find that the magnitude of discontinuity in the soliton mass at $P_0 = 2$ is

$$\Delta M(2, T) = \varepsilon^{s}(0) - \tilde{\varepsilon}^{s}(0) = T \ln[1 + \exp(-1/T)]. \tag{4.34}$$

Finally, we have numerically solved the basic equations for the cases $P_0 = \nu_1 + 1/\nu_2$, $\nu_1 + 0$ and ν_1 and evaluated the coupling constant dependencies of the free energy, shortest-breather mass and soliton mass. Figure 3 shows these quantities at $T = 2$. The results are consistent with our analysis above.

5. Summary and conclusion

In this paper, we have formulated the MTM-SG thermodynamics in the attractive coupling regime and in the zero-charge sector by the BA variational method. We have started with the Bethe wavefunction found by BT and Korepin and constructed the physical vacuum by filling the Dirac sea modes. The elementary excitations upon the physical vacuum are breathers, holes and K excitations and their renormalised energy spectra can be written in terms of the renormalised soliton mass. We have then considered a generic excited state, i.e. a simultaneous excitation of various fundamental objects and imposed PBC on each elementary excitation as well as the Dirac sea modes. This procedure has led to the quantisation of the physical momenta of elementary excitations in terms of renormalised two-body phase shifts as described in (3.22). This equation plays an important role in the BA variational formulation of the MTM thermodynamics. The next step toward the thermodynamics is to express the free energy as a functional of density distributions $\rho_i(\alpha)$ and $\tilde{\rho}_i(\alpha)$ with i representing breathers, holes and K excitations and to minimise it with respect to independent variations of these densities. Thus we reach (4.5) which, along with (4.1), constitute the basic equations for the MTM thermodynamics. By solving these equations one can find densities of elementary excitations, the free energy and hence all the thermodynamic quantities. Moreover, we have shown that the quantities $\varepsilon_j(\alpha)$ for j denoting breathers and holes represent the excitation spectra at finite temperatures. Equations (4.20)-(4.23) are the simplified version of (4.5) and are suitable for actual calculations. These equations can be further simplified in the limit $P_0 = \nu_1 + 0$ as (4.20a, b') and (4.25). On the other hand, for the rational coupling constant $P_0 = \nu_1$ or $\mu = \pi(1 - 1/\nu_1)$ the basic equations are (4.27). By analysing the cases $P_0 = \nu_1$ and $\nu_1 + 0$, we have shown that the free energy is an analytic function of the coupling constant, which is consistent with Araki [17]. We have also found that breather masses are continuous but the soliton mass is discontinuous at the point $P_0 = \nu_1$ due to the dissociation of the longest breather into a soliton-antisoliton pair when P_0 passes ν_1 from larger values. In the analytically soluble case $P_0 = 2 + 0$, i.e. the free MTM limit, (4.34) gives the magnitude of discontinuity in the soliton mass.

We have successfully applied the BA variational method to the MTM-SG thermodynamics. However, as occasionally pointed out in the text, the method contains several ambiguities. It is noted that some of the phase shifts, Δ, have discontinuities at $\alpha = 0$. For example, $\Delta_{11}^{\mathrm{bb}}(\alpha)$ is discontinuous at $\alpha = 0$ (cf (3.14) and (3.16)), and its α derivative contains a δ-function term. In the BA variational method, we always ignore such singular terms without any plausibility arguments. Accordingly, the physical meaning of, say, $\varepsilon_1^{\mathrm{b}}(\alpha)$ as the finite-temperature excitation spectrum of the shortest breather also becomes ambiguous.

In a recent paper, one of us has developed a fermionic perturbation theory for the statistical mechanics of the non-linear Schrödinger model and removed the same ambiguities as above in this model [19]. An application of this approach to the MTM will be reported in a future publication.

Finally, some remarks are due on the discontinuity of the soliton mass. It is first noted that, due to complete integrability, the intrinsic interaction of the MTM does not bring about thermal equilibrium and the existence of a heat bath is essential for reaching thermal equilibrium. In the BA variational formulation, we implicitly assume that interaction with the heat bath realises thermal equilibrium but not finite lifetimes of elementary excitations. However, this is apparently not true; the real situation is a competition between thermal fluctuation and complete integrability. For example, with increasing temperature the soliton mass discontinuity becomes pronounced (cf (4.33)), but at the same time thermal fluctuation becomes stronger and suppresses the singularity. Now a crucial question is whether the thermal fluctuation can completely destroy the singularities arising from complete integrability. At present we do not know an answer to this question, but we can point out the following things.

(i) A satisfactory answer to this question can be obtained only when we go beyond the BA formalism and explicitly consider the interaction between a heat bath and the BA system.

(ii) An affirmative answer to the question would greatly decrease the importance and usefulness of the MTM–SG theory.

Johnson and Fowler have recently discussed the problem of soliton mass discontinuity [20]. They criticised that the ε function defined as in (4.6) does not correspond to the physical soliton mass at finite temperatures. They considered the simple case $\mu=\frac{1}{2}\pi+0$ and argued that the BA wavefunction at $\pi/2$ which contains only solitons and antisolitons and that at $\pi/2+0$ which contains a zero-binding-energy breather are physically indistinguishable. Thus the physically measured soliton mass must be continuous whereas the ε function is discontinuous. In making this comparison they have implicitly assumed the existence of an arbitrarily small non-integrable perturbation H' in the physical measurement so that they can rearrange the zero-binding-energy breather as a pair of solitons and antisolitons.

Because they have taken the limit $\mu\rightarrow\pi/2$ before setting $H'\rightarrow 0$, the physical mass as a function of H' and μ is continuous in μ for any small but non-zero value of H'. However, if one takes the limit $H'\rightarrow 0$ first and if one ignores the thermal fluctuation effect as described above, then our $\varepsilon_s(0)$ should correspond to the soliton mass and is a discontinuous function in μ. As long as one is considering the completely integrable system ($H'=0$) the definition of $\varepsilon_s(0)$ as the soliton mass is unambiguous (see derivations (4.8)-(4.14)).

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Appendix

Here we derive (3.19) from (3.10*b*). The problem is the fourth term on the right of (3.10*b*):

$$I=\int_{-\infty}^{\infty}\mathrm{d}\alpha\, F_j(\alpha)\frac{\partial}{\partial\alpha}\Phi_i(\alpha,\alpha_i). \tag{A1}$$

Remember that the ith K excitation has length $n=n_i=1+\mathrm{i}\nu_1$. Introduce the Fourier transform of $F_j(\alpha)$:

$$\tilde{F}_j(y)=\int_{-\infty}^{\infty}\mathrm{d}\alpha\,\exp(-\mathrm{i}\alpha y)F_j(\alpha).$$

Similarly the Fourier transform of $\phi(\alpha, \alpha_i)$ is written as $\tilde{\phi}(y, \alpha_j)$ and that of $\Phi_j(\alpha, \alpha_i)$ as $\tilde{\Phi}_j(y, \alpha_j)$. Then the Fourier transform of (2.30) is

$$2\pi\left(1+\frac{\sinh(\pi-2\mu)y}{\sinh \pi y}\right)\tilde{F}_j(y)=\tilde{\Phi}_j(y, \alpha_j)-A(n_j)\tilde{\phi}(y, \alpha_i). \tag{A2}$$

As for the Fourier transform of the G function as given by (2.22):

$$\begin{aligned}\tilde{G}_k(y, \alpha_j) &= \frac{1}{\mathrm{i}y}\int_{-\infty}^{\infty} \mathrm{d}\alpha \exp(-\mathrm{i}\alpha y)\frac{\sin Q}{\cosh(\alpha-\alpha_j)-\cos Q} \\ &= \frac{2\pi}{\mathrm{i}y}\exp(-\mathrm{i}\alpha_j y)\frac{\sinh(\pi-\langle Q\rangle)y}{\sinh \pi y}\end{aligned} \tag{A3}$$

where $Q=k\mu-(n_j-B_j)\pi$ and $\langle Q\rangle \equiv Q-2\pi[Q/2\pi]$ with [ ] denoting Gauss' symbol. With the use of (A2) and (A3), (A1) becomes

$$\begin{aligned} I &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}y\,\tilde{\Phi}_i(y, \alpha_i) \\ &\quad\times\left(A(n_i)\exp(\mathrm{i}\alpha_j y)-\exp(\mathrm{i}\alpha_j y)\frac{\sinh(\pi-\langle Q_j^+\rangle)y+\sinh(\pi-\langle Q_j^-\rangle)y}{\sinh \pi y+\sinh(\pi-2\mu)y}\right) \\ &= A(n_j)\Phi_i(\alpha_j, \alpha_i)-\pi A(n_i)A(n_j)\varepsilon(\alpha_j-\alpha_i)-2\pi A(n_i)F_j(\alpha_i)+C \end{aligned} \tag{A4}$$

where $Q_j^{\pm}=(n_j\pm 1)\mu-(n_j-B_j)\pi$. Note that the first and third terms in (A4) cancel the second and third terms in (3.10*b*) and

$$\begin{aligned} C &= -\int_{-\infty}^{\infty}\frac{\mathrm{d}y}{\mathrm{i}y}\exp[\mathrm{i}(\alpha_i-\alpha_j)y]\sinh\frac{2\cosh \mu y}{\sinh \pi y\sinh(\pi-\mu)y} \\ &\quad\times\sinh\left(\pi-\frac{\langle Q_i^+\rangle+\langle Q_i^-\rangle}{2}\right)y\sinh\left(\pi-\frac{\langle Q_j^+\rangle+\langle Q_j^-\rangle}{2}\right)y. \end{aligned} \tag{A5}$$

The integration in (A5) becomes a contour integration by an addition of the integration along the upper semicircle in the complex y plane. There are two types of poles inside this contour: $y=\mathrm{i}k$ and $k\pi\mathrm{i}/\omega$, $k=1,2,\ldots$. After a straightforward calculation, we find that the first-type poles contribute

$$-\sum_{l_j}\sum_{l_i}\phi(\beta_{l_j}-\beta_{l_i}) \tag{A6}$$

which cancels the first term on the right of (3.10*b*). On the other hand, the second-type poles contribute

$$\begin{aligned} -\sum_{l=-(M_i-1)}^{M_i-1} -\mathrm{i}\ln\Bigg(&\frac{\sinh\{\pi[\alpha_i-\alpha_j+\mathrm{i}\omega(n_i+n_j)-\mathrm{i}\pi(l+M_j+1)]/2\omega\}}{\sinh\{\pi[\alpha_i-\alpha_j+\mathrm{i}\omega(n_i+n_j)+\mathrm{i}\pi(l+M_j+1)]/2\omega\}} \\ &\times\frac{\sinh\{\pi[\alpha_i-\alpha_j+\mathrm{i}\omega(n_i+n_j)-\mathrm{i}\pi(l+M_j+1]/2\omega\}}{\sinh\{\pi[\alpha_i-\alpha_j-\mathrm{i}\omega(n_i+n_j)+\mathrm{i}\pi(l+M_j-1)]/2\omega\}}\Bigg) \end{aligned} \tag{A7}$$

which is − (the right-hand side of (3.19)). The remaining second term on the right of (A4) is cancelled by a jump in the *integer* of (3.9) when α_i passes α_j. In the scattering of a hole with the ith K excitation, (2.28) tells us that such a jump is $A(\eta_i)$. Applying a similar argument to (3.3), one can show that the discontinuity is now $A(n_i)A(n_j)$ and precisely cancels the second term on the right of (A4).

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